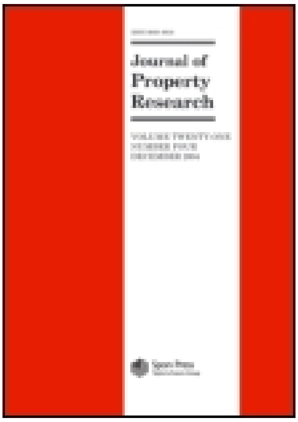

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Diversification effect of real estate investment trusts: Comparing copula functions with kernel methods

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Diversification effect of real estate investment trusts: comparing copula functions with kernel methods

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Value at Risk estimated with joint distribution methodologies demonstrates that risk is lower for portfolios of real estate investment trusts (REITs) and small-business equities compared with a single-asset holding. Benefits from diversification were largest in 2001–2003 and the smallest from 2006–2008.

Previous research using Value at Risk points out the importance of model selection. Various estimation approaches affected results modestly over the entire period (1989–mid 2008). The Value at Risk is -3.1% for two copula models and -3.2% for a nonparametric empirical joint density, at a 1% probability level for weekly returns. After June 1996, the nonparametric copula model consistently returned the lowest risk estimate among the three joint distribution methods.

Time-varying risk is a more important driver in the results than model specification. The highest portfolio risk was found for the period after August 2006 (weekly losses of 4.4% to 5%). The distribution-based model results were closer to the undiversified model results than in the earlier time periods, which supports the premise that contagion across asset classes characterises the post-2006 real estate bust, but is not a strong characteristic of the market over a longer investment horizon that includes growth phases of the business cycle.

Keywords: Value at Risk; REITs; copula; portfolio diversification; structural break tests

1. Introduction

With the expansion of the modern real estate investment trust (REIT) in the 1990s, investors became increasingly able to add a wider choice of property assets to their portfolio while maintaining liquidity and favourable tax treatment. Property assets are expected to provide a defensive component to a stock market portfolio under typical economic conditions. This expectation was challenged recently as the residential housing market declined and subprime mortgage derivatives collapsed, precipitating the financial crisis of 2008.

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Portfolio risk analysis techniques deserve closer study given the boom and bust cycles of the US market (Greenspan, 2008; McDonald, 2009; Salmon, 2009). In this research, portfolio risk and the effects of diversification with REITs are measured with Value at Risk. We pay particular attention to the robust identification of structural changes so that the position of REITs within broader capital markets is represented as accurately as possible. We select a copula approach to estimation of portfolio Value at Risk (PVaR). PVaR is defined as the risk exposure represented by the left tail of a portfolio's probability distribution, hence an estimation of the joint density estimated is warranted. VaR became widely used for financial market risk analysis after J.P. Morgan's publication of the RiskMetrics Technical Document in 1995, followed by several applied studies (Basak & Shapiro, 2001; Christofferson, Hahn & Inouie, 2001; Linsmeier & Pearson, 2000; Jorion, 2001, 2002; Mattedi, Ramos, Rose, & Mantegna, 2004; Yamai & Yoshiba, 2005). VaR is now the standard risk measure set by the Basle Committee on Banking Supervision (1995, 2005) for internal models to determine market risk capital levels. The popularity of VaR is attributed to its simplicity and for regulatory reasons.

The PVaR analyses provide a contrast among three distribution-based approaches – a nonparametric kernel estimation model, a two-stage parametric copula model, and a two-stage nonparametric copula model. All these approaches permit relaxation of the normality and symmetric independence assumptions that have been identified as problematic in some PVaR estimation approaches. The copula-based density estimator is appealing as the copula allows the separability of dependence structures and individual marginal distributions. Moreover, the two-stage copula-based model avoids the potential problem of boundary bias that is sometimes associated with one-stage estimation of joint distributions using kernel smoothing techniques (Silverman, 1986).

In addition to the methodological contribution provided in this study, we update the empirical findings on structural breaks in the REIT market and the time-varying dependence of the REIT and equity assets in the portfolio. As a result, the findings capture the interaction of the two assets during cycles in the capital markets. In a downturn, the REITs and the Russell 2000 equities demonstrated significant co-movement, or asymmetric dependence, that was captured by the PVaR estimates through its focus on the lower tail of the joint distribution. According to these results, there is no single 'best' copula functional form, but a two-parameter copula is consistently the better fit among the parametric copulas. Overall, our findings support the traditional investment premise that REITs contribute to diversification of an equity portfolio, even though within a shorter time frame of a market downturn, diversification benefits are found to be more limited.

2. Literature

Portfolio diversification is a compelling rationale for investing in real estate assets. Earlier studies of property securities suggested a defensive role for REITs, finding that the connection of REITs to equities markets has declined over time, thereby lending credence to the diversification rationale for investing in REITs. Khoo, Hartzell, and Hoesli (1993) found that beta coefficients for equity REITs decreased during the 1980s. Liang, MacIntosh and Webb (1995) studied the 1973–1989 period, identifying a break point in 1983–1984 and likewise found the relationship to be declining over time. Anderson, Clayton, MacKinnon & Sharma (2005) reported a significant sector-specific component underlying REIT returns during 1993–2003, in a multi-asset factor model that includes both REITs and private real estate assets. Lee and Stevenson (2007) identified a structural break in the early 1980s between REITs and other investments including bonds, and, further, found little correlation between REITs and equities. Low correlation of REITs with small-cap stocks (measured with the Russell 2000 index) and large-cap stocks have been documented in the literature, including Ghosh, Miles and Sirmans (1996), Clayton and MacKinnon (2001), Glascock, Lu, and So (2000), and Ziering, Winograd, and McIntosh (1997). The empirical findings on the growing disconnection between REITs and equities market during these periods suggested potential for risk reduction when REITs are added to stock portfolios. These studies of REIT-equity-bond market relationships typically took a time series regression approach on expected returns, with limited attention to risk.

Subsequent studies using GARCH-type specifications, which formally include volatility in the model, find that time-varying conditional correlations between REITs and equities increased during the 2000s (Case, Yang, & Yildirim 2010; Chiang, Kozhevnikov, Lee, and Wisen 2006; Cotter & Stevenson, 2006). The literature provides arguments that the changing relationship between REITs and equities may arise from several sources. Some studies note the changing legal/tax status of REITs in the USA and model these as distinct regimes (with the modern REIT era beginning after August 1991) (Case et al., 2010, among others). The S&P index began to include some REITs after 9 October 2001, another institutional change that has been found to affect co-movement of REITs and equities (Ambrose, Lee & Peek 2007).

Several conventional approaches for PVaR estimation are currently put into practice. For instance, the standard undiversified PVaR is estimated by the sum of the identified tail loss amounts for each asset in the portfolio. The variance-covariance method is based on the assumption that the joint return distribution follows multivariate normality and the PVaR is the loss that equals or exceeds α percent of the time. Furthermore, simulation-based PVaR methodologies are widely used (Han, Pek & Han, 2007; Siegl & West, 2001). However, the Monte Carlo simulation approach has a potential problem, which is that the assumed distribution may not accurately describe the actual distribution of market factors. Usually, the actual distributions of changes in market rates of return have fat tails with respect to the normal distribution (Goorah, 2007). Therefore, an approach that does not impose normality is important to avoid understating risk. A limitation of VaR relates to the coherence criteria established by Artzner, Delbaen, Eber, and Heath (1999) particularly with respect to aggregation. Studies by Danielsson and de Vries (2003), Odening and Hinrichs (2003), and McNeil (1999) found that an extreme value approach to estimate VaR is preferable to historical simulation. The Hill estimator was found to be appropriate for modelling fat-tailed distributions in studies by Zivot and Wang (2003), Danielsson and de Vries (2003) and McNeil (1999).

The understanding of non-normal distributions remains an area of intense interest, especially in the multivariate mixed asset portfolio case. Up to the present, one may use either a one-stage estimation of the joint distribution, conducted with nonparametric methods, or a two-stage estimation using a copula function. Nonparametric distribution estimation, making no assumptions about the distribution form, is able to capture the stylised pattern of the actual distribution. Some research, including Pagan and Ullah (1999) and Li and Racine (2006), supports the use of nonparametric approaches.

Copula-based estimation is capable of generating joint distributions from the marginal distributions and of imposing the desired degree of dependence among the market factors according to Nelsen (1999) and Chen and Huang (2007). The copula approach is particularly useful to identify the joint-tail distribution in order to represent the probabilities of systemic risk or contagion across asset types. For example, Charpentier and Segers (2007) analysed and refined the Archimedean copulas for multivariate distributions. Moreover, copula-based estimations for PVaR are drawing increasing attention in the literature (Embrechts, Höing, & Juri, 2003; Clemente & Romano, 2003; Knight, Lizieri, & Satchell, 2005; Miller & Liu, 2006; Goorah, 2007).

Within the literature on copulas, there are several options based on either parametric or nonparametric approaches as reported in Knight et al. (2005), Poon, Rockinger, and Tawn (2004), Kole, Koedijk, and Verbeek (2007), Okimoto (2008), Goorah (2007) and Rong and Truck (2010). Poon et al. (2004) used the Gumbel parametric form of copula to compare with the Hill estimator (Hill, 1975), while Kole's and Rong and Truck's results supported the student's *t* parametric form. In terms of empirical findings, these studies suggest increasing dependence of equities markets globally and linkages between REITs and equities, leading to the conclusion of contagion rather than beneficial diversification.

3. Methodology

3.1. Value at Risk

Value at Risk (VaR) is defined as the loss over a specific horizon within a given confidence level. For example, consider a portfolio containing two assets, X_1 and X_2, with a joint distribution function F_{X_1,X_2}. The portfolio $VaR_\alpha(X_1 + X_2)$, namely PVaR, is just the α quantile of F_{X_1,X_2}. By selecting α at .01 or .05, the PVaR estimate represents the loss associated with an outcome in the left tail, which occurs with 1% or 5% probability. The tail loss level is multiplied by the value of assets in the portfolio to obtain PVaR in dollars.

Modelling the relationship among different asset returns (e.g. joint distribution) is critical in estimating the PVaR of multivariate portfolios. However, two salient features that frequently characterise asset returns, deviation from normality and asymmetric dependence, make it a challenge to model the joint distribution of asset returns.

The copula function is a method to join or couple multivariate distribution functions to their one-dimensional marginal distribution functions. Due to the separability of dependence structure and marginal distribution functions, a copula-based estimator can accommodate rich dependence structures with few parameters, and allow for marginals to be estimated nonparametrically such that estimation of dependence structure is robust to misspecification of the marginal. Besides, such an estimator can model features such as tail dependence, to capture stylised properties of the tail distribution. This study will utilize different copula estimation approaches to compare parametric and nonparametric forms. Both copula estimates utilise a two-stage approach that provides appropriate control of the separate dependence structures.

As an alternative to copula estimation, the empirical joint density is estimated with a single-stage kernel density estimator, given an optimal bandwidth. Any dependence structures in this alternative empirical estimation are modelled only in combined form. Some dependence structures are captured well with this empirical density method, particularly for symmetric distributions.

3.2. Alternative estimators of portfolio VaR

This study provides a contrast of PVaR among three distribution-based approaches: including a nonparametric (kernel smoothing) model, a two-stage parametric copula model and a two-stage nonparametric copula model. The bivariate nonparametric distribution estimator is widely used for non-normal distributions. (See the review by Scott and Sain (2005).) Suppose a portfolio consists of two assets denoted by X_1 and X_2. The estimated bivariate distribution function is given by:

$$\hat{F}(x_1, x_2) = \frac{1}{n}\sum_{i=1}^{n} G(\frac{x - X_i}{\hat{h}}) = \frac{1}{n}\sum_{i=1}^{n} \int_{-\infty}^{\frac{x_1 - X_{1i}}{\hat{h}_1}} \int_{-\infty}^{\frac{x_2 - X_{2i}}{\hat{h}_2}} k(t_1)k(t_2) \cdot dt_1 dt_2 \qquad (1)$$

where G is the kernel distribution function, k is the kernel density function, and $\hat{h}$ denotes the optimal bandwidth computed by the Least Square Cross-Validation (LS-CV) estimator. See more details in Li and Racine (2006. pp. 24–28). Choosing ‘optimal’ bandwidths is crucial in the kernel estimation as it influences the degree of smoothness of the estimated density. For example, the estimated density might be under-smoothed (over-smoothed) if a small (large) value of bandwidth is chosen. Several bandwidth selection methods, such as Rule-of-Thumb, Plug-in, Likelihood Cross-Validation, and LS-CV, have been well developed for nonparametric kernel estimation. The Rule-of-Thumb and Plug-in methods need additional assumptions about the underlying distribution, whereas the latter two methods are data-driven. Since the Likelihood Cross-Validation method may lead to inconsistent results for fat-tailed distributions, we use the LS-CV method for the selection of bandwidth.

According to Sklar’s theorem (1996), every joint distribution can be written in terms of a copula and its univariate marginal distributions. That is, $C(F_1(x_1), F_2(x_2)) = F(x_1, x_2)$. Given the potential advantage of copulas, researchers prefer incorporating the copula function into the multivariate distribution estimation rather than directly estimating the joint distributions (Patton, 2006; Patton, 2009). We adopt the following procedure laid out in Joe (1997) to conduct the two-stage parametric copula estimation for PVaR.

Step 1: Estimate the distributions of two assets denoted by $\hat{u}$ and $\hat{v}$ such that

$$\hat{u} = \hat{F}_1(x_1) = \frac{1}{n}\sum_{i=1}^{n} \int_{-\infty}^{\frac{x_1 - X_{1i}}{\hat{h}_1}} k(t)dt \text{ and } \hat{v} = \hat{F}_2(x_2) = \frac{1}{n}\sum_{i=1}^{n} \int_{-\infty}^{\frac{x_2 - X_{2i}}{\hat{h}_2}} k(t)dt, \qquad (2)$$

where $\hat{h}_1$ and $\hat{h}_2$ are the optimal bandwidths evaluated by LS-CV for the estimation of the distribution of x_1 and x_2 respectively.

Step 2: Select an appropriate copula function among a copula function family to best fit the data based on the likelihood value. A two-parameter family of copula is adopted due to its flexibility of including rich dependence structures.

For example, the BB1 copula function, $C(u,v) = \{1 + [(u^{-\theta} - 1)^{\delta} + (v^{-\theta} - 1)^{\delta}]^{\frac{1}{\delta}}\}^{-\frac{1}{\theta}}$, where θ and δ are the parameters to be estimated, is found to fit the data best among a two-parameter family of copulae for the analysis of the full dataset. The other BB functional forms that best fit the data in different time periods are presented in the results section.

Step 3: Estimate the parameters of copula function by maximum likelihood methods. For the example in step 2, the estimated copula function is given by:

$$\hat{C}(\hat{u},\hat{v}) = \left\{1 + [(\hat{u}^{-\hat{\theta}} - 1)^{\hat{\delta}} + (\hat{v}^{-\hat{\theta}} - 1)^{\hat{\delta}}]^{\frac{1}{\hat{\delta}}}\right\}^{-\frac{1}{\hat{\theta}}} \tag{3}$$

Step 4: Calculate portfolio VaR given the weights of each asset in a portfolio. If two assets are equally weighted, the 1% PVaR is equal to the sum of 50% of a_1 and 50% of a_2, where a_1 and a_2 are the corresponding real values of returns of two assets such that the estimated $\hat{C}(\hat{u},\hat{v})$ equals 1%.

The procedure to conduct the two-stage nonparametric copula estimation is similar to that for the two-stage parametric copula estimation. The only difference is in step 2. That is, the copula function is estimated nonparametrically, i.e.: $\hat{C}(u_1,v_1) = \frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{\frac{u_1-\hat{u}}{\tilde{h}_1}}\int_{-\infty}^{\frac{v_2-\hat{v}}{\tilde{h}_2}} k(t_1)k(t_2)\cdot dt_1 dt_2$, where $\tilde{h}_1$ and $\tilde{h}_2$ are the optimal bandwidths jointly estimated based on $\hat{u}$ and $\hat{v}$ that are the parameters of the distribution.

For all three distribution-based PVaR estimators, there is an infinite number of combinations of corresponding returns of two assets, a_1 and a_2, at which the estimated $\hat{C}(\hat{u},\hat{v})$ equals 1% (5%). The numerical methods employed ensure that the estimates focus on the lower quantile of the joint density.

3.3. System approach to detect multiple structural breaks at unknown dates

Association between asset returns likely changes over time due to evolution in market conditions and/or regulations. For this reason, we investigate the potential structural breaks in asset markets in addition to estimating PVaR for the entire period of study. We adopt an approach that was recently proposed by Qu and Perron (2007) to deal with multiple structural changes occurring at unknown dates in a system of equations. This system approach has at least two advantages over alternative approaches. First, changes that result from either the regression coefficients and/or the covariance matrix of the errors are identified with this procedure. Second, restrictions on these parameters to analyse (1) partial structural change models, (2) common breaks occurring in all equations, or (3) breaks occurring in a subset of equations are feasible. The flexibility of this approach is an advantage over many other methods such as those in Bai and Perron (1998). Furthermore, this system approach provides several statistical tests to validate each structural break.

We leave the details to Qu and Perron (2007), but offer a brief presentation below. The estimation method is quasi-maximum likelihood based on normally distributed errors. Let (y_t, x_t) characterise the system of equations. Conditional on a given partition of the sample $T = (T_1, \ldots, T_m)$, the Gaussian quasi-likelihood function is:

$$L_T(T, \beta, \Sigma) = \prod_{j=1}^{m+1} \prod_{t=T_{j-1}+1}^{T_j} f(y_t | x_t; \beta_j, \Sigma_j) \tag{4}$$

such that $f(y_t|x_t;\beta_j,\Sigma_j) = \frac{1}{(2\pi|\Sigma_j|)^{1/2}} \exp\{-\frac{1}{2}[y_t - x_t'\beta_j]'\Sigma_j^{-1}[y_t - x_t'\beta_j]\}$, where β_j are coefficients and Σ_j is the variance-covariance matrix. The quasi-likelihood ratio is

$$LR_T = \frac{\prod_{j=1}^{m+1} \prod_{t=T_{j-1}+1}^{T_j} f(y_t|x_t;\beta_j,\Sigma_j)}{\prod_{j=1}^{m+1} \prod_{t=T_{j-1}^0+1}^{T_j^0} f(y_t|x_t;\beta_j^0,\Sigma_j^0)} \tag{5}$$

The aim is to obtain values of ($T_1,\ldots,T_m$, β, Σ) that maximise LR_t subject to the restrictions $g(\beta, vec(_\Sigma)) = 0$. Let $lr_T(\ )$ denote the log likelihood ratio and $rlr_T(\ )$ denote the restricted log likelihood ratio. The objective function: rlr_T $(T,\beta,\Sigma) = lr_T$ $(T,\beta,\Sigma) + \lambda' g(\beta, vec(\Sigma))$ and the estimates are $(\hat{T}, \hat{\beta}, \hat{\Sigma}) = \text{argmax}_{(T1,Tm;\beta;\Sigma)} rlr\text{-}T(T, \beta, \Sigma)$.

4. Data

REITs are an investment mechanism that provides real estate holdings with certain tax advantages and the benefits of being a liquid asset. The FTSE NAREIT all-REIT total return index is the data source. The Russell 2000 index measures the performance of the small-cap segment of the US equity universe. The Russell 2000 is an appropriate series for public equities in this portfolio analysis because the scale of firms is more comparable to REITs than would be the case if REITs were analysed with an index that is dominated by very large corporations. Clayton and MacKinnon (2001) also study REITs against the Russell. To the extent that diversification exists between this equity index and REITs, the reason is more likely to be due to the property sector than as a result of the size of the companies.

Daily REIT and Russell 2000 indices are collected for February 1989 through July 2008. We then transform the daily data to the weekly average index using an equal weight of five days in a week. Aggregation to a weekly frequency represents a reasonable exposure period for a practising risk manager using PVaR. However, due to the requirement of structural change analysis, the first six weekly observations are not included in the study of PVaR. Therefore, the analysis starts from the third week of March 1989. Figure 1 presents the distribution of returns of the two assets over time, across different time periods separated by the identified structural breaks. Overall, the weekly returns of the Russell 2000 have a larger range compared with REITs. To some degree, the returns of these two assets exhibit co-movements according to preliminary visual inspection.

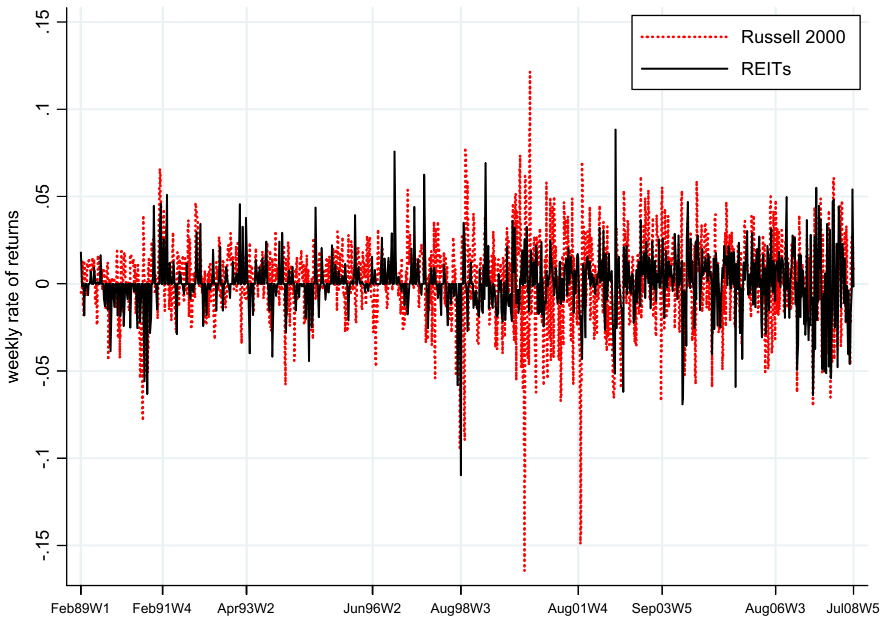


Figure 1. Time series plot of weekly rates of return: REITs and Russell 2000 indices.
Note: The vertical grids divided the entire data between the first week in February of 1989 (Feb89W1) to the last week in July of 2008 (Jul08W5) into eight time periods separated by the identified seven structural breaks using Qu and Perron's system method.

Over the entire time period, 1989–2008, returns on REITs have a smaller average weekly return and lower variation than that of Russell 2000 (Table 1). The correlation of the two series over the entire period is 0.26. However, during some time periods, for example, in periods 2 and 7 (from the first week of March in 1991 till the second week of April in 1993; and from the first week of October in 2003 till the third week of August in 2006), REITs had higher but less volatile returns. Table 1 also shows excessive skewness and kurtosis, especially for REIT returns, which suggests a likely deviation from normality.

5. Results

5.1. Structural breaks and diagnostic results

Seven structural breaks in the rates of return on REITs and Russell 2000 indices appear during 1989–2008. The corresponding week of those identified structural breaks are given by the horizontal axis in Figure 1. Our finding of structural breaks identifies the first break in 1991, somewhat earlier than the findings by Glascock et al. (2000), Ziering et al. (1997) and Clayton and MacKinnon (2001). It is plausible that our findings differ because the portfolio of two assets has different time series properties than the univariate data that was tested in the earlier studies.

There have been two major boom periods in equities markets in the last 20 years that provide a back-drop to the structural change modelling results. First, the advent of internet technology-based firms as popular investments defined a period of generally rising equity market values during the late 1990s. One can point to the initial public offering of Netscape stock on 9 August 1995 as a rough demarcation

Table 1. Summary statistics of weekly returns on REITs and Russell 2000 indices, 1989–2008.

Period (N)	Mean	Min	Max	Std Dev.	Skewness	Kurtosis	Return at certain percentile		
							1%	5%	10%
Rate of return on REITs (percentage)									
1989–2008 (1,010)	.0006	−.1096	.0884	.0165	−.373	8.51	−.051	−.025	−.017
Mar. 1989–Feb. 1991 (100)	−.0033	−.0631	.0462	.0137	−.661	9.81	−.056	−.025	−.018
Feb. 1991–Mar. 1993 (111)	.0029	−.0289	.0512	.0116	1.58	7.85	−.024	−.009	−.005
Apr. 1993–Jun. 1996 (166)	.0000	−.0441	.0438	.0106	−.329	9.42	−.042	−.018	−.009
Jun. 1996–Aug. 1998 (116)	.0006	−.1096	.0760	.0172	−1.347	21.56	−.058	−.017	−.008
Aug. 1998–Aug. 2001 (155)	.0008	−.0269	.0692	.0142	1.003	5.56	−.023	−.018	−.016
Aug. 2001–Sep. 2003 (109)	.0012	−.0619	.0884	.0184	.157	8.07	−.051	−.031	−.022
Sep. 2003–Aug. 2006 (150)	.0030	−.0689	.0470	.0178	−1.302	6.16	−.066	−.029	−.021
Aug. 2006–Jul. 2008 (103)	−.0014	−.0639	.0553	.0268	−.128	2.63	−.054	−.049	−.040
Rate of return on Russell 2000 (percentage)									
1989–2008 (1,010)	.0018	−.1644	.1217	.0244	−.700	7.42	−.063	−.040	−.027
Mar. 1989–Feb. 1991 (100)	.0003	−.0784	.0660	.0210	−.611	5.36	−.054	−.042	−.023
Feb. 1991–Mar. 1993 (111)	.0033	−.0341	.0463	.0165	.227	2.88	−.032	−.024	−.018
Apr. 1993–Jun. 1996 (166)	.0029	−.0578	.0299	.0143	−.817	4.62	−.044	−.024	−.015
Jun. 1996–Aug. 1998 (116)	.0000	−.0938	.0540	.0207	−1.11	6.12	−.054	−.035	−.028
Aug. 1998–Aug. 2001 (155)	.0027	−.1644	.1217	.0345	−.635	6.40	−.089	−.055	−.042
Aug. 2001–Sep. 2003 (109)	.0011	−.1489	.0686	.0316	−1.007	6.50	−.067	−.050	−.035
Sep. 2003–Aug. 2006 (150)	.0025	−.0589	.0590	.0234	−.128	2.95	−.057	−.035	−.026
Aug. 2006–Jul. 2008 (103)	.0004	−.0701	.0608	.0261	−.304	2.99	−.065	−.045	−.036

Note: Figures in parenthesis are total number of observations in each time period.

of the technology boom in equities markets. The technology bubble burst around March 2000. REITs also trended upward during the internet boom but the REITs index declined earlier than technology, and recorded no significant gains on average during 1993–1996. During the 1990s, REIT regulation and tax treatment in the USA changed and, according to some studies, REITs became less correlated with equities. This suggests considerable potential for REITs to contribute to portfolio risk management through diversification.

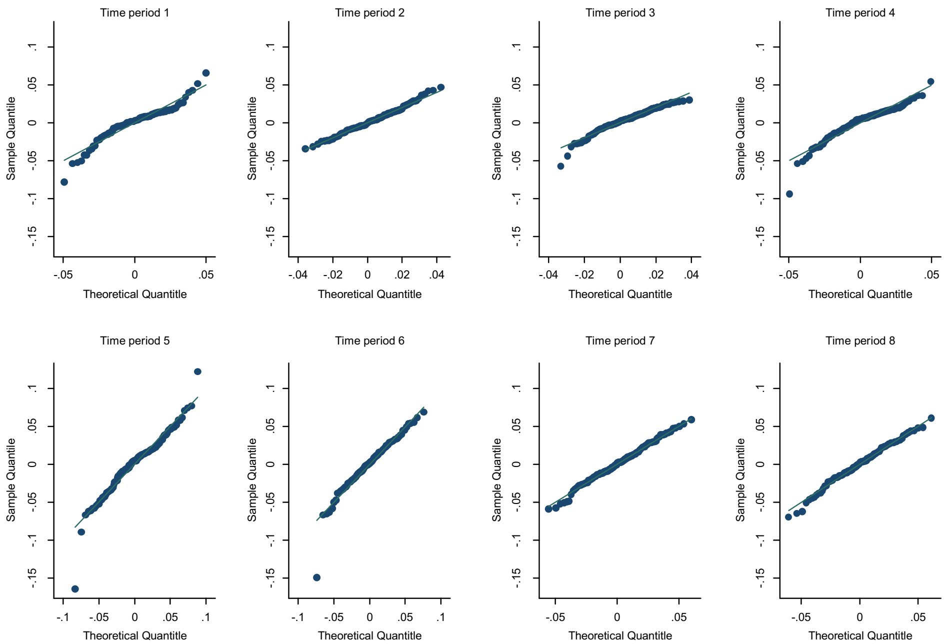
After the technology stocks fell out of favour, the second major boom period in equities had its roots in real estate. Investors turned to properties as well as real-estate funds in search of better returns. Financial services deregulation was well underway and some firms in mortgage lending grew rapidly, contributing to growth in the mortgage REITs component of the NAREIT index. The real estate boom period is reflected in the structural change analysis in October 2003–August 2006 (period 7), when the average weekly return on the all-REIT index reached 0.3%. The mean weekly return for the Russell was 0.25%, also strong.

The period from 2006–2008 signifies the bust period for the real estate market. The NAREIT all-REIT index averaged weekly losses of 0.14%. This downturn is not without precedent. Note that in the real estate bust period that occurred fifteen years earlier, February 1989–February 1991, the losses in the REIT index were even more dramatic, with average weekly returns of −0.33%.

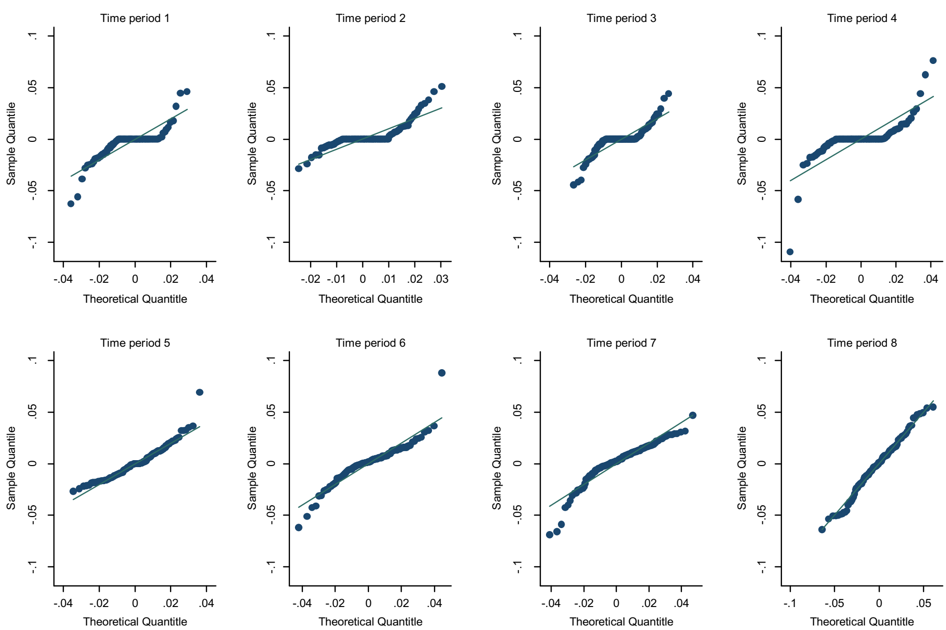
Considerable volatility shifts over time also contributed to the findings on structural breaks. The period from 1989–1998 shows lower volatility compared with the later years. The periods from September 2001–September 2003 and August 2006–July 2008 were both booms and busts in the market and experienced high standard deviations of returns.

The statistical properties of the two series also reveal characteristics of the data separated by those seven structural breaks. Figure 2 shows a quantile–quantile (Q-Q) plot of weekly returns of REITs and Russell 2000 indices indicating departure from the standard normal density. The plot reveals departures from normality, according to the scatter of the points away from the 45-degree line. For example, the rates of return on REITs in each of the first seven periods distinctively depart from normality, but the departure is less obvious in the last period (Figure 2(a)). As shown in Figure 2(b), returns of the Russell 2000 index exhibit a departure from normality as indicated by the non-linear behavior observed in periods 1, 3, 4, 5, and 6, but such departure is not distinctive in periods 2, 7 and 8. Three different univariate normality tests confirm the graphical findings, as do multivariate normality tests (Table 2). The joint normality hypothesis is rejected in the first seven periods and over the entire period at the significance level less than 1%. The invalid joint normality assumption supports the use of nonparametric methods and non-Gaussian copulae to estimate the joint density.

In order for the benefits of portfolio diversification to be apparent with a PVaR indicator, the variables must exhibit low dependence in the negative tail of the distribution. A chi-plot graphical analysis (Abberger, 2005; Fisher & Switzer 1985; Fisher & Switzer 2011) is used to investigate whether covariates are independent or have more complex dependence structure. The value χ_i measures the departure from bivariate statistical independence. At each sample point, χ_i is actually a correlation coefficient between dichotomised X values and dichotomised Y values. Therefore, all values of χ_i lie in the interval $[-1, 1]$. If Y is a strictly increasing (decreasing) function of X, $\chi_i = 1$ ($\chi_i = -1$) for any i. It is common to supplement the basic chi-



(a) REITs' rates of return



(b) Russell 2000's rates of return

Figure 2. Q-Q plots of rates of return on REITs and Russell 2000 indices in different time periods.

plot with a pair of horizontal guidelines at $\pm C_p/\sqrt{n}$, where C_p is selected so that approximately 100p% of the pairs of (λ_i, χ_i) lies between these two horizontal lines

Table 2. Normality tests for weekly returns on REIT and Russell 2000 indices: test statistics and p-values (in parenthesis).

	Univariate tests for normality						
	Rate of return on REITs			Rate of return on Russell 2000			Shapiro–Wilk test for joint normality
Period	Shapiro–Wilk	Shapiro–Francia	Jarque–Bera	Shapiro–Wilk	Shapiro–Francia	Jarque–Bera	
Mar. 1989–Feb. 1991	.831 (.000)	.818 (.000)	216.706 (.000)	.938 (.000)	.930 (.000)	31.807 (.000)	0.857 (.000)
Feb. 1991–Mar. 1993	.835 (.000)	.827 (.000)	154.919 (.000)	.991 (.708)	.993 (.779)	1.015 (.602)	0.895 (.000)
Apr. 1993–Jun. 1996	.829 (.000)	.820 (.000)	288.294 (.000)	.963 (.000)	.961 (.000)	36.571 (.000)	0.855 (.000)
Jun. 1996–Aug. 1998	.666 (.000)	.645 (.000)	1700.479 (.000)	.935 (.000)	.928 (.000)	71.125 (.000)	0.805 (.000)
Aug. 1998–Aug. 2001	.949 (.000)	.945 (.000)	68.247 (.000)	.955 (.000)	.947 (.000)	84.943 (.000)	0.909 (.000)
Aug. 2001–Sep. 2003	.910 (.000)	.897 (.000)	117.290 (.000)	.950 (.000)	.943 (.000)	74.226 (.000)	0.912 (.000)
Sep. 2003–Aug. 2006	.914 (.000)	.910 (.000)	104.699 (.000)	.993 (.659)	.994 (.741)	.424 (.809)	0.954 (.000)

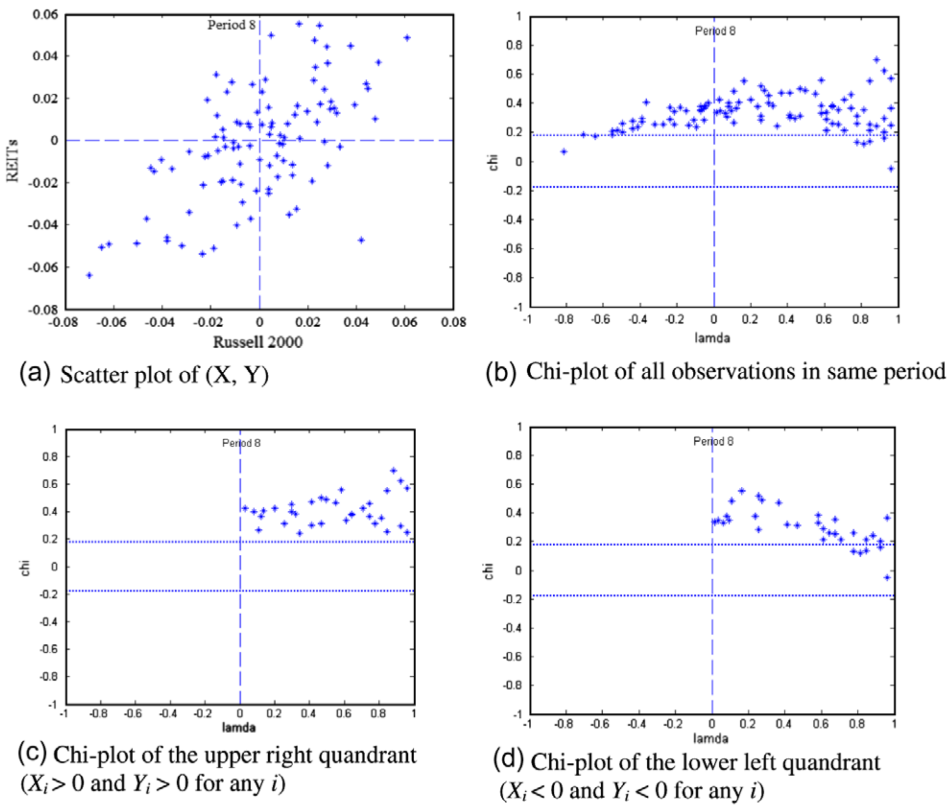
(*Continued*)

Table 2. (*Continued*)

	Univariate tests for normality						
	Rate of return on REITs			Rate of return on Russell 2000			Shapiro–Wilk test for joint normality
Period	Shapiro–Wilk	Shapiro–Francia	Jarque–Bera	Shapiro–Wilk	Shapiro–Francia	Jarque–Bera	
Aug. 2006–Jul. 2008	.985 (.327)	.989 (.492)	.872 (.647)	.990 (.628)	.991 (.625)	1.568 (.457)	0.991 (.871)
1989–2008	.901 (.000)	.899 (.000)	1312.638 (.000)	.956 (.000)	.954 (.000)	911.776 (.000)	0.927 (.000)

if there is no relationship between X and Y. The value λ_i measures the distance of the data point (X_i, Y_i) from their median values. All λ_i 's must lie in the interval $[-1, 1]$. When the data are a random bivariate sample from independent continuous marginals, then the values of the λ_i are individually uniformly distributed. However, when X and Y are associated, the value of the λ_i likely shows clustering. In particular, λ_i is likely to be positive if X and Y are positively correlated, and vice versa for negative correlation.

Figure 3 presents scatter plots and chi-plots for weekly returns of REITs and Russell 2000 indices for the most recent time period (August 2006–July 2008). The majority of the χ_i values lie outside the two horizontal control lines and the values of λ_i show certain clustering instead of uniformly distributed. Such patterns of (λ_i, χ_i) pairs suggests that the returns of REITs and Russell 2000 indices exhibit a complex dependence structure, an asymmetric tail-dependence in this time period. Similarly, the chi-plots for each time period suggest asymmetric dependence in the rate of returns on both REITs and Russell 2000 indices and the dependence structures are time-variant (the chi-plots are available upon request). Given the findings of the invalid normality assumption and the asymmetric tail-dependence, it is important to utilise both the univariate marginal distributions and the dependence structure to characterise the joint distribution of the returns of these two assets.



(a) Scatter plot of (X, Y)

(b) Chi-plot of all observations in same period

(c) Chi-plot of the upper right quandrant ($X_i > 0$ and $Y_i > 0$ for any i)

(d) Chi-plot of the lower left quandrant ($X_i < 0$ and $Y_i < 0$ for any i)

Figure 3. Scatter plot and chi-plots for weekly returns of REITs and Russell 2000 indices for the most recent time period (September 2006–July 2008).

5.2. Copula estimators of portfolio VaR

We employ three data-driven density estimators in the analysis of PVaR: nonparametric, copula-based parametric, and copula-based nonparametric methods. These three approaches provide reliable density estimates regardless of the shape of the underlying distribution. In particular, the asymmetric tail-dependence, revealed in the majority of time periods in our data, can be well captured by the estimators we use. Among these three methods, we particularly concentrate on the copula-based density estimator due to its appealing statistical property – separability of dependence structure and individual marginal distributions. Boundary bias is possible when numerical methods are applied to estimate a distribution across a large domain (Li & Racine, 2006). The two-stage estimation procedure that we use avoids the potential problem by, first, estimating marginal distribution functions separately and then, in the second stage, obtaining an estimate of the joint distribution function in terms of the previously estimated cumulative distribution functions which are bounded on [0, 1].

There is an infinite number of combinations of asset returns at the level of 1% of the joint distribution function. However, as PVaR focuses on the pitfall of asset returns only, combinations such that both individual returns lie lower than the twentieth percentile of its marginal distribution are of interest in this study. Consider an example to illustrate the procedure of estimating the PVaR in time period 6 once the associated joint distribution of two-asset returns is estimated. First, we choose the returns of one asset (Russell 2000 index) that lie in the left tail of its marginal distribution. In particular, we choose the return associated with the 2, 4,..., twentieth percentiles. The values of such returns as well as the corresponding percentile are presented in the first column in Table 3. Given the choices of the asset returns on the Russell 2000 index, we search for the associated returns of the REIT index at which the combination of these two returns has a joint probability of 1%. In the search we exclude the returns of REITs that lie above the twentieth percentile of its marginal distribution. Such returns on REITs as well as the associated percentiles of REITs are provided in column 2 for the nonparametric approach, in column 4 for the two-stage parametric copula approach, and in column 6 in the two-stage nonparametric copula approach. The 5% joint probability results are also shown for comparison purposes (Table 4).

Assuming REITs and Russell 2000 indices are equally weighted, then the 1% PVaR equals the sum of 50% of the corresponding returns on the two assets. Columns 3, 5 and 7 in Table 3 present the 1% PVaR for each combination of two-asset returns using three different distribution-based estimation approaches. For example, the case presented in the first cell of Table 3 suggests the following findings:

(1) The probability that the weekly rate-of-return loss on the portfolio is greater than 0.049 is 1%; and
(2) at such loss, one particular combination of returns is −0.067 for Russell 2000 and −0.041 for REITs, which corresponds to the 2nd and 6.4th percentiles of the distribution of each asset.

The same procedure is applied to estimate the 1% PVaR for other time periods. It can also be adapted if two assets are not equally weighted or a lower percentile (less than 20) is practically more appropriate. Alternative portfolio weighting

Table 3. Illustration of PVaR of equally weighted bivariateution-based methods for time period 6 (September 2001–September 2003).

	Empirical nonparametric		Two-stage parametric copula		Two-stage nonparametric copula	
Russell 2000	REITs	PVaR	REITs	PVaR	REITs	PVaR
−.067	−.026	−.047	−.031	−.049		
(2%)	(6.4%)		(4.6%)			
−.063	−.031	−.047	−.041	−.052	−.014	−.038
(4%)	(4.6%)		(3.7%)		(14.7%)	
−.047	−.041	−.044	−.052	−.049	−.026	−.037
(6%)	(3.7%)		(1.8%)		(6.5%)	
−.038	−.041	−.040	−.052	−.045	−.031	−.034
(8%)	(3.7%)		(1.8%)		(5.7%)	
−.035	−.041	−.038	−.052	−.043	−.031	−.033
(10%)	(3.7%)		(1.8%)		(5.5%)	
−.032	−.043	−.037	−.052	−.042	−.031	−.031
(12%)	(2.8%)		(1.8%)		(5.5%)	
−.030	−.043	−.037	−.052	−.041	−.031	−.030
(14%)	(2.8%)		(1.8%)		(5.5%)	
−.028	−.043	−.035	−.052	−.040	−.031	−.029
(16%)	(2.8%)		(1.8%)		(5.5%)	
−.024	−.043	−.033	−.052	−.038	−.031	−.027
(18%)	(2.8%)		(1.8%)		(5.5%)	
−.021	−.043	−.032	−.052	−.036	−.031	−.026
(20%)	(2.8%)		(1.8%)		(5.5%)	
Average PVaR at 1% of the joint distribution of bivariate assets assuming equal probability of combinations of asset returns at which the joint probability equals 1%.						
	−.039		**−.0434**		**−.0318**	

Notes: Figures in parentheses are the percentile of returns whose values are presented above parentheses. As we calculate the 1% PVaR based on the joint distribution of two assets, it is likely to have the corresponding returns of REITs above the twentieth percentile of its marginal distribution given the returns of Russell 2000 indices presented in the first column. Since VaR focuses on the pitfall of asset returns, we exclude combinations of asset returns if any of them is above the twentieth percentile of its marginal distribution, which leaves some cells of the table blank in the case of the two-stage nonparametric copula approach.

schemes change the results in terms of specific values for the PVaR, but the key findings in terms of larger risk in the noted sub-periods and the differences between PVaR estimates among the methods are maintained when portfolio weights are adjusted from equal weights to 25%–75%.

For each estimation approach, the individual returns of the two assets have significant variations but the 1% PVaR estimates have a lower range across different combinations of quantiles in period 6 (Table 3). The range of the 1% PVaR estimates does not exceed 1.6 percentage points in rate of return. This salient pattern is maintained in the PVaR estimates for other time periods. Therefore, for ease in interpreting the results, we present the average PVaR across different combinations of two-asset returns. In the case of period 6, the average 1% PVaR are −0.039, −0.043 and −0.032 for three alternative distribution-based approaches, respectively (see the last row in Table 3). That is, on average, within the period there is a probability of 1% that the weekly rate-of-return loss on the portfolio is greater than 3.9% under the nonparametric approach, 4.3% for the two-stage parametric copula approach, and 3.8% for the two-stage nonparametric copula approach. In other words, if a portfolio consists of $500 in REIT and $500 in Russell 2000, the average PVaR in dollars will be between $38 and $43 based on three distribution-based methods. Similarly, we calculate the average 1% PVaR of an equally weighted two-asset portfolio under alternative methodologies for estimation of the joint distribution of two assets and present the results in Figure 4. Significant vari-

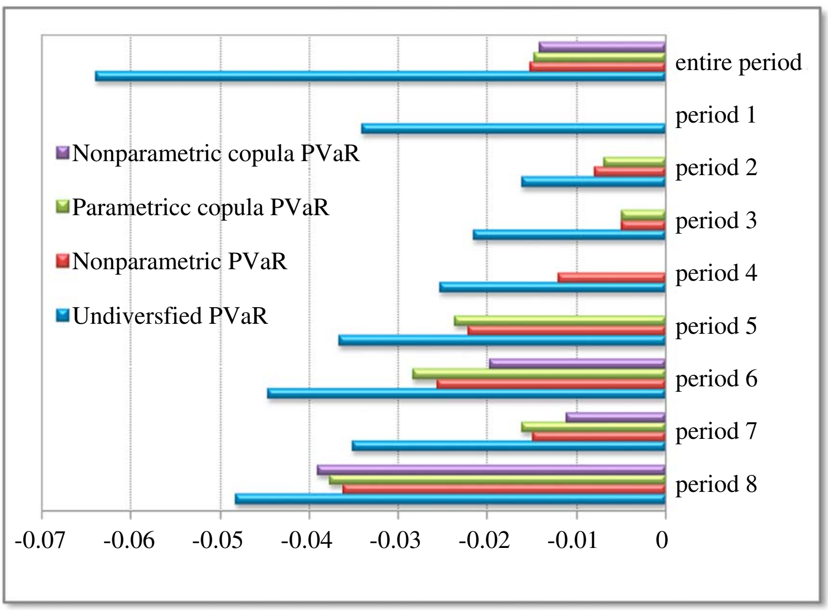


Figure 4. Portfolio value at risk under alternative methodologies for estimations of joint distribution of a two-asset portfolio (REITs and Russell 2000) (1987–2008). Notes: For periods 1–5 under certain approaches, we did not find any combination of two-asset returns that simultaneously meet the following two criteria: (a) the joint distribution of such combinations of returns has a probability of 5%; and (b) the return of each asset is at least at the twentieth percentile of its marginal distribution.

ability in PVaR over time can be observed. Moreover, there is not a consistent pattern of larger or smaller estimates among the alternative copula and kernel methods of estimation.

5.3. Results on portfolio diversification

The findings on diversification are based on the interpretation of PVaR as a risk indicator; there are benefits from diversification if the riskiness of the portfolio is less than the riskiness of an asset held individually. First, we compare the distribution-based PVaR values with the undiversified PVaR (Table 4). The results show that the undiversified PVaR is clearly greater than the other PVaR measures based on the joint distribution of two assets. The overestimation of riskiness of the portfolio is mainly because this particular approach completely ignores the association of the two assets and, thus, considers no diversification impacts. Furthermore, the undiversified PVaR estimator is unable to capture the stylised property of fat tail distribution, and as a result, the undiversified PVaR is not supported by our data.

Figure 5 illustrates the benefits of asset diversification by comparing the distribution-based PVaR values with the VaR for a single asset. Except in period 5 (August 1998–August 2001), the greatest risk measured with the three distribution-based PVaR approaches is smaller than the stand-alone VaR of either REITs or Russell 2000. That is, a portfolio of REITs and Russell 2000 assets has a smaller VaR than a one-asset portfolio, which supports the risk-reduction effects of portfolio diversification. Even in period 5, when REITs had relatively low risk, the diversified portfolio has a smaller PVaR than holding small business assets alone. Clearly, the benefit of diversification using a portfolio including REITs and Russell 2000 is supported. This result may be considered a confirmation of the traditional finding of the diversification benefit of REITs as an asset class.

5.4. Selection of copula functional form and comparison of three PVaR estimation methods

The goodness-of-fit tests based on mean squared error provide a clear preference for the parametric copula over the nonparametric in statistical terms in most of the sub-periods (Table 5). Two exceptions are noteworthy: periods 1 and 4 are particularly problematic in terms of the fit of the parametric copula. As a practical matter, however, the PVaR estimates generally are quite close to each other in any of the distribution-based approaches that we use. The largest differences across models are in August 1998–September 2003, at around 1 percentage point for PVaRs in the 2–4% range.

The PVaR calculated with an empirical joint density is consistently higher than the copula results (in absolute value) after 1991. The nonparametric copula approach provides the lowest risk estimate of the three joint distribution methods in each period after June 1996.

It is not surprising that, for the entire period, the BB1 two-parameter copula is a better fit than the Gumbel, Gaussian or other parametric forms that we tested. Having two parameters to control in the BB parametric form provides a more flexible structure that fits the data better in this application. Different BB-family forms were

Table 4. Portfolio Value at Risk under alternative methodologies for estimation of joint distribution of a two-asset portfolio (1989–2008).

		Distribution-based estimation approaches		
	Undiversified	Empirical nonparametric	Two-stage parametric copula	Two-stage nonparametric copula
		1% probability of joint distribution		
1989–2008	−.0595	−.0324	−.0306 (BB1)	−.0312
Mar. 1989–Feb. 1991	−.0708	−.0235	NA	−.0244
Feb. 1991–Mar. 1993	−.0315	−.0156	−.0129 (BB10)	−.0172
Apr. 1993–Jun. 1996	−.0429	−.0170	−.0151 (BB10)	−.0163
Jun. 1996–Aug. 1998	−.0760	−.0202	NA	−.0140
Aug. 1998–Aug. 2001	−.0582	−.0327	−.0341 (BB8)	−.0234
Aug. 2001–Sep. 2003	−.1054	−.0390	−.0434 (BB9)	−.0318
Sep. 2003–Aug. 2006	−.0617	−.0301	−.0296 (BB8)	−.0265
Aug. 2006–Jul. 2008	−.0670	−.0494	−.0500 (BB1)	−.0435
		5% probability of joint distribution		
1989–2008	−.0330	−.0152	−.0147 (BB1)	−.0141
Mar. 1989–Feb. 1991	−.0340	NA	NA	NA
Feb. 1991–Mar. 1993	−.0161	−.008	−.0068 (BB10)	NA
Apr. 1993–Jun. 1996	−.0215	−.0050	−.0049 (BB10)	NA
Jun. 1996–Aug. 1998	−.0254	−.0120	NA	NA
Aug. 1998–Aug. 2001	−.0367	−.0221	−.0237 (BB8)	NA
Aug. 2001–Sep. 2003	−.0447	−.0257	−0284 (BB9)	−.0198
Sep. 2003–Aug. 2006	−.0351	−.0149	−.0162 (BB8)	−.0111
Aug. 2006–Jul. 2008	−.0483	−.0362	−.0380 (BB1)	−.0390

Notes: The symbol, NA, implies that during the certain period using a particular approach, we do not find any combination of two-asset returns that simultaneously meet the following two criteria: (a) the joint distribution of such combinations of returns has a probability of 1% (5%); and (b) the return of each asset is at least at the twentieth percentile of its marginal distribution. The fitted parametric copula function for the two-stage parametric copula approach is denoted in the parenthesis for each time period that fits the data best among the two-parameter copula function families given the goodness of fit based on mean squared error.

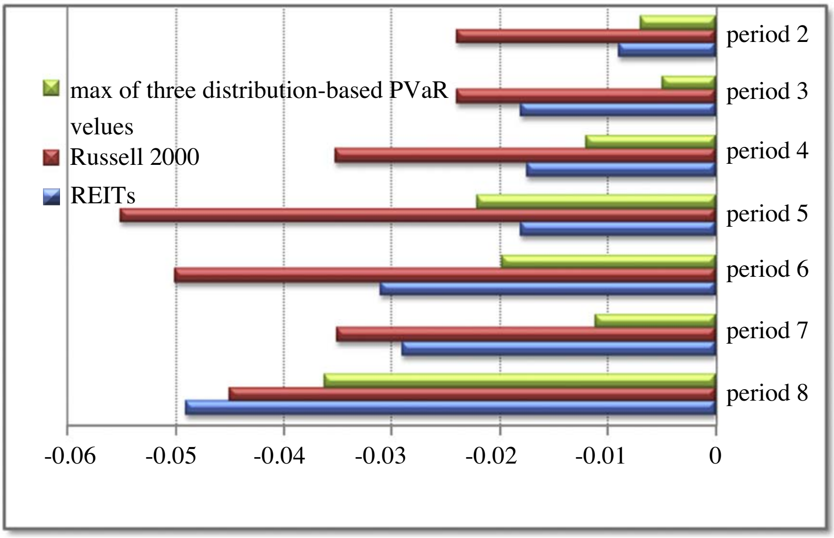


Figure 5. Diversification effects in different time periods. Notes: For periods 1 using any of the three distribution-based approaches, we did not find any combination of two-asset returns that simultaneously meet the following two criteria: (a) the joint distribution of such combinations of returns has a probability of 5%; and (b) the return of each asset is at least at the twentieth percentile of its marginal distribution.

appropriate in the sub-periods, as shown in Table 6. Each of the BB-copulas is a two-parameter variant of a copula expression that is established in the literature.

However, there was no solution to the measure of systemic risk in terms of copula estimate of the joint density in several sub-periods that we examined. The inability to obtain valid numerical estimates is a greater problem for VaR estimates at the 5% joint probability than for the lower tail 1% probability as a result of the larger number of possible combinations.

The PVaR values under three alternative distribution-based approaches exhibit some time-variant attributes (Figure 5). The risk exposure is greater in recent times (period 8, when the 1% PVaR on a $1,000 portfolio is between $43 and $50 based on three approaches), followed by period 6 ($32–43), period 7 ($27–30), and then

Table 5. Results of goodness-of-fit tests based on mean squared error.

	Two-stage parametric copula	Two-stage nonparametric copula
	Mean Squared Error	
1989–2008	0.1600124	0.76815219
Mar 1989–Feb 1991	14.28280077	0.19875275
Feb 1991–Mar 1993	0.02307371	0.76456456
Apr 1993–Jun 1996	0.02328876	0.18094616
Jun 1996–Aug 1998	14.89878156	0.37376186
Aug 1998–Aug 2001	0.05633691	0.35490831
Aug 2001–Sep 2003	0.23674800	0.34096880
Sep 2003–Aug 2006	0.03584341	0.10771339
Aug 2006–Jul 2008	0.03155858	0.03970184

Table 6. Bivariate two-parameter copula families used in two stage parametric model.

Period	Bivariate copula function $C(u, v; \theta, \delta)$	$(\hat{\theta}, \hat{\delta})$
Entire	BB1: $\{1 + [(u^{-\theta} - 1)^{\delta} + (v^{-\theta} - 1)^{\delta}]^{1/\delta}\}^{-1/\theta}$	(0.1181, 1.1742)
1	BB7: $1 - (1 - [(1 - \bar{u}^{\theta})^{-\delta} + (1 - \bar{v}^{\theta})^{-\delta} - 1]^{-1/\delta})^{1/\theta}$	(1.3776, 0.0050)
2	BB10: $uv[1 - \theta(1 - u^{1/\delta})(1 - v^{1/\delta})]^{-\delta}$	(0.5218, 0.1398)
3	BB10: $uv[1 - \theta(1 - u^{1/\delta})(1 - v^{1/\delta})]^{-\delta}$	(0.9950, 14.7860)
4	BB6: $1 - (1 - \exp\{-[(-\log(1 - \bar{u}^{\theta}))^{\delta} + (-\log(1 - \bar{v}^{\theta}))^{\delta}]^{1/\delta}\})^{1/\theta}$	(1.0246, 1.1046)
5	BB8: $\delta^{-1}[1 - \{1 - [1 - (1 - \delta)^{\theta}]^{-1}[1 - (1 - \delta u)^{\theta}][1 - (1 - \delta v)^{\theta}]\}^{1/\theta}]$	(2.1955, 0.6313)
6	BB9: $\exp\{-[(\delta - \log u)^{\theta} + (\delta - \log v)^{\theta} - \delta^{\theta}]^{1/\theta} + \delta\}$	(1.8479, 0.0499)
7	BB8: $\delta^{-1}[1 - \{1 - [1 - (1 - \delta)^{\theta}]^{-1}[1 - (1 - \delta u)^{\theta}][1 - (1 - \delta v)^{\theta}]\}^{1/\theta}]$	(13.6671, 0.1655)
8	BB1: $\{1 + [(u^{-\theta} - 1)^{\delta} + (v^{-\theta} - 1)^{\delta}]^{1/\delta}\}^{-1/\theta}$	(0.8457, 1.2599)

Note: $\bar{u} = 1 - u$ and $\bar{v} = 1 - v$.

period 5 ($23–34). The least risk was found for periods 2 ($13–17) and 3 ($15–17). These findings confirm that the main driver in Value at Risk is the selection of input data rather than the specific details of the methodologies in the distribution-based models. Under this circumstance, a range of historical data along with appropriate structural break estimation methods helps to identify the worst case combination of asset movements without ignoring the benefits of portfolio diversification.

6. Conclusions

The rates of return on real estate and small cap equities exhibit non-normality and asymmetric dependence, leading to the use of three data-driven distribution-based approaches to estimate the joint distribution and then to estimate the portfolio risk and diversification. Namely, a nonparametric kernel smoothing model, a two-stage parametric copula model, and a two-stage nonparametric copula model are used to accommodate distributions of any shape as well as asymmetric dependence between assets. The copula-based density estimator is potentially more appealing as it allows the separability of dependence structure and individual marginal distributions.

The empirical application of these three distribution-based approaches is a portfolio of real estate investment trusts and small business equities. This application has the following main findings. First, for each estimation approach, the returns of the two assets have significant variations but the portfolio value at risk (PVaR) estimates are fairly close to each other based on three distribution-based approaches when the impacts of structural breaks are considered. Consider a portfolio consisting of REITs and Russell 2000 with equal weights. Take the PVaR in period 6 as an example. There is a probability of 1% that the weekly loss on the portfolio is greater than 3.9% under the nonparametric kernel smoothing approach, 4.3% for the two-stage parametric copula approach and 3.8% for the two-stage nonparametric copula approach. In other words, if a portfolio consists of $500 in REIT and $500 in Russell 2000, the average 1% PVaR in dollars is between $38 and $43 based on three distribution-based methods.

Second, these two financial asset indexes are shown to provide diversification benefits in terms of risk reduction, using PVaR as the risk indicator. In particular, the undiversified PVaR is clearly greater than the other alternative PVaR measures based on the joint distribution of two assets as it completely ignores the association of two assets. Furthermore, the maximum of PVaR values estimated using these three distribution-based approaches is smaller than the stand-alone VaR of either REITs or Russell 2000. That is, a portfolio of REITs and Russell 2000 assets has a smaller VaR than a one-asset portfolio, which supports portfolio diversification. Establishing a portfolio of shares of REITs and Russell 2000 indexes provides greater diversification benefits than ownership of an individual asset.

The similar PVaR values based on three distribution-based approaches accompanied by different PVaR values across time periods confirm that the main driver in this risk management tool is the selection of input data, rather than the specific details of the methodologies in the distribution-based models. We recommend the use of the best structural break estimation methods as the basis for stress tests, so that the worst case combination of asset movements is well represented without ignoring the benefits of portfolio diversification. Structural breaks identified with a time-series system comprising multiple portfolio elements are demonstrated here to be effective in isolating boom and bust periods relevant to the portfolio. Because the system approach to structural break tests includes both the portfolio components as well as autocorrelation patterns, considerable information on the changes in the periods is taken into account. The resulting PVaR estimates for the identified break periods are possible inputs into models of systemic risk or worst-case scenarios that can provide a useful basis for financial decision making.

The use of copula functions in the context of portfolio Value at Risk is not without limitation. We find difficulties in fitting a two-parameter copula to certain subperiods of the study timeframe and the difficulties were more common when the 5% significance level for PVaR was selected instead of 1%. This suggests the need to explore a variety of distribution-based methods, including the emerging method of copula-extreme value theory. Furthermore, in addition to the selection of different assets in a portfolio, asset weights are another potential management tool to reduce portfolio risk. It is a direction for future research to modify the current methodologies in estimating the joint density given a framework of minimising portfolio risk by choosing the asset weights of a portfolio.

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